Indian IMO Team Selection Tests 1994

First Test

May 18

1. A vector in the coordinate plane initially coinciding with OM_0 rotates about the origin O at a constant speed of $2\pi/n$ radian per second, where $n \in \mathbb{N}$. Let $M_1, M_2, \dots, M_{n-1}$ be the positions of M_0 at the end of $1, 1+2, \dots, 1+2+\cdots+(n-1)$ seconds. Determine the set of values of n for which $M_0, M_1, M_2, \dots, M_{n-1}$ are the vertices of a regular n-gon (in some order).

2. Find all functions $f : \mathbb{N} \to \mathbb{N}$ with the following properties:

   (i) f is strictly increasing;

   (ii) $f(2n) = f(n) + n$ for each $n \in \mathbb{N}$;

   (iii) Whenever $f(n)$ is a prime, n is a prime;

3. Show that the numbers $1, 2, 3, \dots, 1994$ can be colored using 4 colors so that no arithmetic progression of 10 terms has all its members colored the same.

4. A nonisosceles trapezium $ABCD$ with $AB \parallel CD$ and $AB > CD$ possesses an incircle with center I which touches CD at E. Let M be the midpoint of AB and let MI meet CD at F. Show that $DE = FC$ if and only if $AB = 2CD$.

Second Test

May 21

1. Suppose the set $\mathbb{Q}^+$ is partitioned into three disjoint subsets A, B, C satisfying the conditions $BA = B$, $B^2 = C$, $BC = A$, where HK stands for the set $\{hk \mid h \in H, k \in K\}$ for any two subsets H, K of $\mathbb{Q}^+$ and H^2 stands for HH.

2. Show that all positive rational cubes are in A.

3. Find such a partition $\mathbb{Q}^+ = A \cup B \cup C$ with the property that for no positive integer $n \le 34$ are both n and $n+1$ in A; that is,
$$\min\{n \in \mathbb{N} \mid n \in A, n+1 \in A\} > 34.$$

4. Find the largest integer $n \le 1994$ for which 2^{10} divides $\binom{4n}{n}$.

5. Show that there are infinitely many polynomials P with integer coefficients such that $P(0) = 0$ and $P(x^2 - 1) = P(x)^2 - 1$.

6. In the triangle ABC, let D, E be points on the side BC such that $\angle BAD = \angle CAE$. If M, N are, respectively, the points of tangency with BC of the incircles of the triangles ABD and ACE, show that
$$\frac{1}{MB} + \frac{1}{MD} = \frac{1}{NC} + \frac{1}{NE}.$$

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Third Test

May 25

1. Let $ABCD$ be a convex quadrilateral such that the internal angle bisectors of the four angles of the quadrilateral form a nondegenerate convex quadrilateral $PQRS$. Prove that the Euler circles of the triangles QRS, RSP, SPQ, PQR have a point in common.

2. Suppose that $\mathscr{F}$ is a family of k-element subsets of a $2k$-element set X such that each $(k-1)$-element subset of X is contained precisely in one member of $\mathscr{F}$. Show that $k+1$ is a prime number.

3. Let a_k and b_k, $1 \le k \le n$, be positive real numbers. Show that
$$\sqrt[n]{a_1 a_2 \cdots a_n} + \sqrt[n]{b_1 b_2 \cdots b_n} \le \sqrt[n]{(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)}.$$

4. Two distinct real numbers a and b are such that $a^n - b^n$ is an integer for each $n = 1, 2, \ldots$. Show that a and b are themselves integers.

Fourth Test

May 28

1. In triangle ABC with $AB = AC$, D is the foot of the altitude from C, E the midpoint of CD, and F the foot of the perpendicular from A to BE. Let K be the foot of the perpendicular from A to CF. Prove that $AK \le \frac{1}{3}AB$.

2. Find all positive integers n for which there exists a permutation $(a_1, a_2, \ldots, a_n)$ of $1, 2, \ldots, n$ such that i divides $a_1 + a_2 + \cdots + a_i$ for each i, $1 \le i \le n$.

3. Show that in any sequence of length 2^n of n symbols $a_1, a_2, \ldots, a_n$ there exists a block in which each symbol occurs an even number (possibly zero) times. Furthermore, show that this conclusion in not necessarily true for a sequence of length $2^n - 1$.

4. A person starts at the origin and makes a sequence of moves along the real axis with k-th move being a change of $+k$ or $-k$.

   (a) Prove that the person can reach any integer.

   (b) If $M(n)$ is the least number of moves required to reach a positive integer n, prove that
   $$2 - \frac{1}{\sqrt{n}} < \frac{M(n)}{\sqrt{n}} < \sqrt{2} + \frac{3}{\sqrt{n}}.$$

Fifth Test

May 29

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1. The incircle of a triangle ABC touches the sides BC, CA, AB at D, E, F, respectively. Let P be any point within the incircle and let the segment AP, BP, CP meet the incircle at points X, Y, Z, respectively. Prove that the lines DX, EY, FZ concur.

2. A finite set $a_1, a_2, \ldots, a_n$ of positive integer is called *good* if a_i divide $a_1 + a_2 + \cdots + a_n$ for each $i = 1, \ldots, n$. Prove that every finite set of positive integer is contained in some good set.

3. If $a_1, a_2, \ldots, a_n$ are positive numbers, prove the inequality

$$\sum_{j=1}^{N} \sqrt[j]{a_1 a_2 \cdots a_j} < 3 \sum_{j=1}^{N} a_j.$$

4. Let $N(S)$ denote the number of subsets of a finite set S (i.e. $N(S) = 2^{|S|}$). Suppose that A, B, C are finite sets with $|A| = |B| = 1994$ and $N(A) + N(B) + N(C) = N(A \cup B \cup C)$. Determine the minimum positive value of $|A \cap B \cap C|$. Give an instance where this minimum is realized.

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