13-th Indian Mathematical Olympiad 1998

1. Let AB be a chord of a circle $\mathscr{C}_1$ that is not a diameter, and M be the midpoint of AB. Let T be a point on the circle $\mathscr{C}_2$ with OM as diameter. The tangent to $\mathscr{C}_2$ at T meets $\mathscr{C}_1$ at P. Show that
$$PA^2 + PB^2 = 4PT^2.$$

2. Let a and b be positive rational numbers. Prove that if $\sqrt[3]{a} + \sqrt[3]{b}$ is a rational number, then so are $\sqrt[3]{a}$ and $\sqrt[3]{b}$.

3. Let p,q,r,s be four integers with $5 \nmid s$. If there is an integer a for which $pa^3 + qa^2 + ra + s$ is divisible by 5, prove that there is an integer b such that $sb^3 + rb^2 + qb + p$ is also divisible by 5.

4. A convex quadrilateral $ABCD$ is inscribed in a circle of unit radius. Show that if $AB \cdot BC \cdot CD \cdot DA \geq 4$, then $ABCD$ is a square.

5. Suppose a,b,c are real numbers such that the quadratic equation
$$x^2 - (a+b+c)x + (ab+bc+ca) = 0$$
has roots of the form $\alpha \pm i\beta$, where $\alpha > 0$ and $\beta \neq 0$ are real numbers. Show that:
   (a) The numbers a,b,c are all positive.
   (b) The numbers $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are the sides of a triangle.

6. We want to choose n of the $2n$ integers $0,0,1,1,2,2,\dots,n-1,n-1$ such that the average of the n chosen integers is an integer and as small as possible. Show that this can be done for each positive integer n and find this smallest value.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
Typed in LaTeX by Ercole Suppa
www.imomath.com