6-th Indian Mathematical Olympiad 1991

1. Find the number of positive integers n such that

   (i) $n \leq 1991$,

   (ii) $n^2 + 3n + 2$ is a multiple of 6.

2. In an acute-angled triangle ABC, the altitude from A meets the semicircle with diameter BC constructed outwards at point A'. Points B' and C' are defined analogously. Prove that
$$S_{BCA'}^2 + S_{CAB'}^2 + S_{ABC'}^2 = S_{ABC}^2,$$
where S_{XYZ} denotes the area of triangle XYZ.

3. Given a triangle ABC, denote
$$\begin{aligned} x &= \tan\frac{B-C}{2}\tan\frac{A}{2}, \\ y &= \tan\frac{C-A}{2}\tan\frac{B}{2}, \\ z &= \tan\frac{A-B}{2}\tan\frac{C}{2}. \end{aligned}$$
Prove that $x + y + z + xyz = 0$.

4. Let a, b, c be real numbers in the interval $(0,1)$ with $a + b + c = 2$. Prove that
$$\frac{a}{1-a} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c} \geq 8.$$

5. In a triangle ABC with incenter I, points X, Y are taken on the segments AB, AC respectively such that $BX \cdot AB = IB^2$ and $CY \cdot AC = IC^2$. Given that the points X, I, Y are collinear, find the possible values of $\angle A$.

6. (a) Find all positive integers n for which 3^{n+1} divides $2^{3^n} + 1$.

   (b) Prove that 3^{n+2} does not divide $2^{3^n} + 1$ for any positive integer n.

7. Determine all real solutions x, y, z of the system
$$\begin{cases} x + y - z = 4, \\ x^2 - y^2 + z^2 = -4, \\ xyz = 6. \end{cases}$$

8. We are given 10 objects of integer weights with the total weight 20. Prove that if none of the weights exceds 10, then the objects can be divided into two groups of equal weights.

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9. The incircle l of a triangle ABC is centered at I and touches the side BC at T. The line through T parallel to IA meets the incircle again at S and the tangent to the incircle at S meets AB, AC at points C', B', respectively. Prove that the triangle $AB'C'$ is similar to the triangle ABC.

10. For any positive integer n, let $s(n)$ denote the number of ordered pairs (x,y) of positive integers for which
$$\frac{1}{x}+\frac{1}{y}=\frac{1}{n}.$$
Determine all those n for which $s(n)=5$.

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