25-th Indian Mathematical Olympiad 2010

First Day

1. Let Γ be the circumcircle of $\triangle ABC$. Let M be a point in the bisector of $\angle A$ that is inside $\triangle ABC$. Denote by A_1, B_1, and C_1 the intersection points of AM, BM, and CM with Γ, respectively. Prove that $PQ \| BC$.

2. Find all natural numbers $n > 1$ such that n^2 does not divide $(n-2)!$.

3. Find all non-zero real numbers x, y, z such that the following system of equations is satisfied:

$$\begin{aligned}(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2) &= xyz\\ (x^4+x^2y^2+y^4)(y^4+y^2z^2+z^4)(z^4+z^2x^2+x^4) &= x^3y^3z^3.\end{aligned}$$

Second Day

4. How many 6-tuples $(a_1,a_2,a_3,a_4,a_5,a_6)$ all of which elements are from $\{1,2,3,4\}$ satisfy the following condition: *All of* $a_j^2 - a_ja_{j+1} + a_{j+1}^2$ *for* $j = 1, \dots, 6$ *are mutually equal*?

5. Let ABC be an acute-angled triangle with altitude AK. Let H be its orthocenter and O its circumcenter. Assume that $\triangle KOH$ is acute-angled and that P is its circumcenter. Let Q be the reflection of P with respect to HO. Prove that Q lies on the line joining the midpoints of AB and AC.

6. Define a sequence $(a_n)_{n\geq 0}$ by $a_0 = 0$, $a_1 = 1$, and:

$$a_n = 2a_{n-1} + a_{n-2},$$

for $n \geq 2$.

(a) Prove that $2a_m$ divides $a_{m+j} + (-1)^j a_{m-j}$ for all $m > 0$ and $j \in \{0,1,2,\dots,m\}$.

(b) If $2^k \mid n$ for some natural numbers n and k, prove that $2^k \mid a_n$.

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