24-th Indian Mathematical Olympiad 2009

First Day

1. Let ABC be a triangle and let P be an interior point such that $\angle BPC = 90^\circ$, $\angle BA_{=}\angle BCP$. Let M and N be the midpoints of the segments AC and BC respectively. If $BP = 2PM$, prove that A, P, and N are collinear.

2. Define the sequence $(a_n)_{n=1}^{\infty}$ in the following way:

$$a_n = \begin{cases} 0, & \text{if the number of positive divisors of } n \text{ is odd,} \\ 1, & \text{otherwise.} \end{cases}$$

Prove that $x = 0.a_1a_2a_3\dots$ is irrational.

Remarks. 1 and n are positive divisors of n. x can be understood as $x = \sum_{n=0}^{\infty} a_n \cdot \frac{1}{10^n}$.

3. Find all real numbers x such that:

$$[x^2 + 2x] = [x]^2 + 2[x].$$

Second Day

4. All the points of the plane are colored using three colors. Prove that there exists a triangle with vertices of the same color such that either which either is isosceles or its angles are in geometric progression.

5. Let H be an orthocenter of acute-angled triangle ABC. Denote by h_{max} the largest altitude of $\triangle ABC$. Prove that

$$AH + BH + CH \leq 2h_{max}.$$

6. Let a, b, and c be real numbers such that $a^3 + b^3 = c^3$. Prove that

$$a^2 + b^2 - c^2 > 6(c-a)(c-b).$$

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com