22-nd Indian Mathematical Olympiad 2007

1. In a triangle ABC with a right angle at C, the median through B bisects the angle between BA and the bisector of $\angle B$. Prove that $\frac{5}{2} < \frac{AB}{BC} < 3$.

2. Let a,b,c be natural numbers and $a^2+b^2+c^2=n$. Prove that there exist constants p_i,q_i,r_i $(i=1,2,3)$ independent of a,b,c such that
$$(p_1a+q_1b+r_1c)^2+(p_2a+q_2b+r_2c)^2+(p_2a+q_2b+r_2c)^2=9n.$$
Further, if a,b,c are not all divisible by 3, show that $9n$ can be expressed as $x^2+y^2+z^2$ for some natural numbers x,y,z not divisible by 3.

3. The equation $x^2-mx+n=0$ has real roots α and β, where m and n are positive integers. Prove that α and β are integers if and only if $[m\alpha]+[m\beta]$ is a perfect square.

4. Let $\sigma=(\sigma_1,\dots,\sigma_n)$ be a permutation of $1,\dots,n$. A pair of indices (i,j) is an *inversion* of σ if $i<j$ and $\sigma_i>\sigma_j$. How many permutations of $1,\dots,n$ $(n\geq 3)$ have exactly two inversions?

5. In a triangle ABC with $AB=AC$, D is the midpoint of BC, P a point on AD, and E the orthogonal projection of P on AC. If $AP/PD=BP/PE=\lambda$, $BD/AD=m$ and $z=m^2(1+\lambda)$, prove that
$$z^2-(\lambda^3-\lambda^2-2)z+1=0.$$
Deduce that $\lambda\geq 2$ with the equality if and only if $\triangle ABC$ is equilateral.

6. If x,y,z are positive numbers, prove the inequality
$$(x+y+z)^2(yz+zx+xy)^2\leq 3(y^2+yz+z^2)(z^2+zx+x^2)(x^2+xy+y^2).$$

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