17-th Indian Mathematical Olympiad 2002

1. For a convex hexagon $ABCDEF$ in which each pair of opposite sides is unequal, consider the following six statements:

$$(a_1)\ AB \parallel DE; \quad (b_1)\ BC \parallel EF; \quad (c_1)\ CD \parallel FA;$$
$$(a_2)\ AE = BD; \quad (b_2)\ BF = CE; \quad (c_2)\ CA = DF.$$

   (a) Show that if all the six statements are true, then the hexagon is cyclic.

   (b) Prove that any five of these statements also imply that the hexagon is cyclic.

2. If a,b,c are positive integers, find the least positive value of the expression $a^3 + b^3 + c^3 - 3abc$, as well as all triples (a,b,c) for which this value is attained.

3. Let x,y be positive numbers with $x+y=2$. Prove that $x^3y^3(x^3+y^3) \le 2$.

4. Do there exist 100 lines in the plane, no three of them concurrent, such that they intersect exactly in 2002 points?

5. Do there exist three distinct positive numbers a,b,c such that the numbers

$$a,b,c,b+c-a,c+a-b,a+b-c,a+b+c$$

   form an arithmetic progression of 7 terms in some order?

6. The numbers $1,2,\dots,n^2$ are arranged in an $n\times n$ array such that the numbers in each row and each column are in increasing order. Denote by a_{jk} the number in j-th row and i-th column. For each $1 \le j \le n$, let b_j be the number of possible entries that can occur as a_{jj} (For example, if $n=3$, the numbers that can occur as a_{22} are $4,5,6$, so $b_2=3$). Prove that

$$b_1+b_2+\cdots+b_n \le \frac{n}{3}(n^2-3n+5).$$

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