16-th Indian Mathematical Olympiad 2001

1. For any point P in a non-right triangle ABC, denote by A_1, B_1, C_1 the reflections of P in the sides BC, CA, AB, respectively. Prove that:

   (a) If P is the incenter or an excenter of the triangle, then it is the circumcenter of $\triangle A_1B_1C_1$.

   (b) If P is the circumcenter of the triangle, then it is the orthocenter of $\triangle A_1B_1C_1$.

   (c) If P is the orthocenter of the triangle, then it is either the incenter or an excenter of $\triangle A_1B_1C_1$.

2. Show that the equation
$$x^2+y^2+z^2=(x-y)(y-z)(z-x)$$
has infinitely many solutions in integers x, y, z.

3. Prove that if a, b, c are positive numbers with $xyz = 1$, then
$$a^{b+c}b^{c+a}c^{a+b} \le 1.$$

4. Show that among any nine integers it is possible to choose four, denoted a, b, c, d, such that $a+b-c-d$ is divisible by 20. Show that such a selection may not be possible if there are eight integers instead of nine.

5. Let ABC be a triangle and D be the midpoint of side BC. Suppose that $\angle DAB = \angle BCA$ and $\angle DAC = 15^\circ$. Show that $\angle ADC$ is obtuse. Moreover, if O is the circumcenter of $\triangle ADC$, prove that triangle AOD is equilateral.

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition
$$f(x+y) = f(x)f(y)f(xy) \quad \text{for all } x, y \in \mathbb{R}.$$

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