15-th Indian Mathematical Olympiad 2000

1. The incircle of triangle ABC touches sides BC, CA, AB at K, L, M respectively. The line through A parallel to LK meets MK at P, and the line through A parallel to MK meets LK at Q. Prove that the line PQ bisects AB and AC.

2. Find all integers x, y, z satisfying
$$x + y = 1 - z \quad \text{and} \quad x^3 + y^3 = 1 - z^3.$$

3. If a, b, c, x are real numbers such that $abc \neq 0$ and
$$\frac{xb + (1-x)c}{a} = \frac{xc + (1-x)a}{b} = \frac{xa + (1-x)b}{c},$$
prove that $a = b = c$.

4. In a convex quadrilateral $PQRS$, $PQ = RS$, $(\sqrt{3} + 1)QR = SP$ and $\angle RSP - \angle SQP = 30^\circ$. Prove that $\angle PQR - \angle QRS = 90^\circ$.

5. Prove that if λ is a (real or complex) root of the cubic equation $x^3 + ax^2 + bx + c = 0$ whose coefficients satisfy $1 \geq c \geq b \geq a \geq 0$, then $|\lambda| \leq 1$.

6. For every natural number $n \geq 3$, let $f(n)$ denote the the number of pairwise noncongruent triangles with integer sides and perimeter n. Prove that

   (a) $f(1999) > f(1996)$;

   (b) $f(2000) = f(1997)$.

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