14-th Iberoamerican Mathematical Olympiad

La Habana, Cuba, September 11–19, 1999

First Day

1. Find all natural numbers not that are less than 1000 and have the following property: the cube of the sum of digits of the number is equal to the square of the number itself.

2. We say that circle M bisects circle N if their common chord is a diameter of N. Consider two non-concentric circles C_1 and C_2.

   (a) Prove that there exist infinitely many circles B that bisect both C_1 and C_2.

   (b) Find the locus of the centers of such circles B.

3. On a line in the plane are given $n \geq 2$ distinct points $P_1, P_2, \dots, P_n$. Each of the circles with the diameters P_iP_j $(1 \leq i, j \leq n)$ is colored in one of k given colors. We call such a configuration an (n,k)-*cloud*.

   For each positive integer k, determine all n with the property that every (n,k)-cloud contains two externally tangent circles of the same color.

Second Day

4. Let B be an integer greater than 10 consisting of digits $1,3,7,9$. Show that B has a prime divisor not smaller than 11.

5. An acute-angled triangle ABC is inscribed in a circle with center O. Let AD, BE, CF be its altitudes. The line EF intersects the circle at P and Q.

   (a) Prove that OA is perpendicular to PQ.

   (b) Prove that if M is the midpoint of BC, then $AP^2 = 2AD \cdot OM$.

6. Let A and B be given points in the plane and C be a point on the perpendicular bisector of AB. The sequence $C_1 = C, C_2, C_3, \dots$ of points is defined as follows: if C_n is not on the segment AB, then C_{n+1} is the circumcenter of triangle ABC_n. Find all the positions of C for which C_n is defined for all n and is periodic starting with some term.

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