12-th Iberoamerican Mathematical Olympiad

Guadalahara, Mexico, September 14–21, 1997

First Day

1. A real number $r \geq 1$ has the following property: For any positive integers m and n, m divides n if and only if $[mr]$ divides $[nr]$. Prove that r is an integer.

2. A circle centered at the incenter I of a triangle ABC meets all three sides of the triangle: side BC at D and P (with D nearer to B), side CA at E and Q (with E nearer to C), and side AB at F and R (with F nearer to A). The diagonals of the quadrilaterals $EQFR$, $FRDP$, and $DPEQ$ meet at S, T, and U, respectively. Show that the circumcircles of the triangles FRT, DPU and EQS have a single point in common.

3. For an integer $n \geq 2$, let D_n be the set of points (x,y) of the plane with integer coordinates such that $-n \leq x,y \leq n$.

   (a) Each of the points of D_n is colored with one of three given colors. Prove that there always exist two points of D_n of the same color such that the line passing through them contains no other point of D_n.

   (b) Give an example of a coloring of points of D_n with four colors in such a manner that if a line contains exactly two points of D_n, then these two points have different colors.

Second Day

4. Let n be a positive integer. Consider the sum $x_1y_1 + x_2y_2 + \cdots + x_ny_n$ for any $2n$ numbers a_i, b_i taking only the values 0 and 1. Denote by $I(n)$ the number of $2n$-tuples $(x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_n)$ for which this sum is odd, and by $P(n)$ the number of those for which this sum is even. Prove that
$$\frac{P(n)}{I(n)} = \frac{2^n+1}{2^n-1}.$$

5. In a triangle ABC, AE and BF are altitudes and H the orthocenter. The line symmetric to AE with respect to the bisector of $\angle A$ and the line symmetric to BF with respect to the bisector of $\angle B$ intersect at a point O. The lines AE and AO meet the circumcircle of $\triangle ABC$ again at M and N, respectively. The lines BC and HN meet at P, BC and OM at R, and HR and OP at S. Prove that $AHSO$ is a parallelogram.

6. Let $\mathscr{P} = \{P_1, P_2, \dots, P_{1997}\}$ be a set of 1997 points inside the unit circle with center at P_1. For each $k = 1,2,\dots,1997$, let x_k be the distance from P_k to the closest point in $\mathscr{P}$ different from P_k. Prove that
$$x_1^2 + x_2^2 + \cdots + x_{1997}^2 \leq 9.$$

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