9-th Iberoamerican Mathematical Olympiad

Fortaleza, Brazil, September 17–25, 1994

First Day – September 20

1. A natural number n is called *sensible* if there is an integer r with $1 < r < n-1$ such that the representation of n in base r has all the digits equal. For example, 62 and 15 are sensible, because 62 is written 222 in base 5 and 15 is 33 in base 4. Show that 1993 is not sensible, but 1994 is sensible.

   (Mexico)

2. Let $ABCD$ be a cyclic quadrilateral. Suppose that there exists a circle with center in AB that is tangent to the other sides of the quadrilateral.

   (a) Prove that $AB = AD + BC$.

   (b) Determine, in terms of $x = AB$ and $y = CD$, the maximum possible area of the quadrilateral. *(Brazil)*

3. In each cell of an $n \times n$ chessboard there is a lamp. When a lamp is touched, the state of this lamp and all the lamps in its row and its column is changed. Initially all the lamps are off. Show that it is always possible to turn on all the lamps with finitely many touches, and find in terms of n the minimum number of touches needed. *(Brazil)*

Second Day – September 21

4. An acute-angled triangle ABC is inscribed in a circle k. For a point P inside circle k, the lines AP, BP, CP meet k again at X, Y, Z. Determine the point P for which triangle XYZ is equilateral. *(Brazil)*

5. Let n and r be two given positive integers. We wish to construct r subsets $A_1, A_2, \dots, A_r$ of $\{0, 1, \dots, n-1\}$, each of cardinality k, such that for each integer x with $0 \le x \le n-1$ there exist elements $x_i \in A_i$ $(i = 1, \dots, n)$ with $x = x_1 + x_2 + \cdots + x_r$. Find the minimum value of k. *(Brazil)*

6. Show that all natural numbers $n \le 2^{1000000}$ can be obtained beginning at 1 with less than 1100000 sums, that is, there exists a finite sequence of natural numbers $x_0 = 1, x_1, \dots, x_k = n$ with $k < 1100000$, such that for each $i = 1, 2, \dots, k$, there exist $0 \le r, s < i$ with $x_i = x_r + x_s$. *(Brazil)*

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