5-th Iberoamerican Mathematical Olympiad

Valladolid, Spain, September 22–30, 1990

First Day

1. A function f is defined on the nonnegative integers as follows:

$$f(n) = \begin{cases} 0, & \text{if } n = 2^j - 1,\ j = 0,1,2,\dots; \\ f(n-1) - 1, & \text{otherwise.} \end{cases}$$

   (a) Show that for every n there is an integer $k \geq 0$ such that $f(n) + n = 2^k - 1$.

   (b) Calculate $f(2^{1990})$.

2. In a triangle ABC, I is the incenter and D, E, F the tangency points of the incircle with the sides BC, CA, AB, respectively. The line AD intersects the incircle again at P. If M is the midpoint of EF, prove that the points P, I, M and D lie on a circle.

3. Let $f(x) = (x+b)^2 - c$, where b and c are integers.

   (a) If p is a prime number such that c is divisible by p, but not by p^2, show that p^2 does not divide $f(n)$ for any integer n.

   (b) Let $q \neq 2$ be a prime divisor of c. If q divides $f(n)$ for some integer n, prove that for every positive integer r there exists an integer n' for which q^r divides $f(n')$.

Second Day

4. Let C_1 be a circle, AB its diameter, t the tangent at B, and $M \neq A$ a variable point on C_1. A circle C_2 is tangent to C_1 at M and to the line t.

   (a) Find the tangency point of C_2 and t and find the locus of the center of C_2 as M varies.

   (b) Prove that there is a circle orthogonal to all the circles C_2.

5. A and B are two opposite corners of an $n \times n$ board ($n \geq 1$) divided into n^2 unit squares. Each square is divided into two triangles by a diagonal parallel to AB, giving $2n^2$ triangles in total. A piece moves from A to B going along the sides of the triangles and, whenever it moves along a segment, it places a seed in each of the triangles having that segment as a side. The piece never moves along the same segment twice. It turns out that after the trip every triangle contains exactly two seeds. For which values of n is this possible?

6. Let $f(x)$ be a cubic polynomial with rational coefficients. Prove that if the graph of f is tangent to the x-axis, then all roots of $f(x)$ are rational.

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