21-st Iberoamerican Mathematical Olympiad

Guayakuil, Ecuador, September 22–30, 2006

First Day – September 26

1. In a scalene triangle ABC with $\angle A = 90^\circ$, the tangent line at A to its circumcircle meets line BC at M and the incircle touches AC at S and AB at R. The lines RS and BC intersect at N, while the lines AM and SR intersect at U. Prove that the triangle UMN is isosceles.

2. For n real numbers $a_1, a_2, \dots, a_n$, let d denote the difference between the greatest and smallest of them and $S = \sum_{i<j} |a_i - a_j|$. Prove that

$$(n-1)d \le S \le \frac{n^2}{4}d$$

   and find when each equality holds.

3. The numbers $1, 2, \dots, n^2$ are written in the squares of an $n \times n$ board in some order. Initially there is a token on the square labelled with n^2. In each step, the token can be moved to any adjacent square (by side). At the beginning, the token is moved to the square labelled with the number 1 along a path with the minimum number of steps. Then it is moved to the square labelled with 2, thenj to square 3, etc, always taking the shortest path, until it returns to the initial square. If the total trip takes N steps, find the smallest and greatest possible values of N.

Second Day – September 27

4. Find all pairs (a, b) of positive integers such that $2a-1$ and $2b+1$ are coprime and $a+b$ divides $4ab+1$.

5. The sides AD and CD of a tangent quadrilateral $ABCD$ touch the incircle φ at P and Q, respectively. If M is the midpoint of the chord XY determined by φ on the diagonal BD, prove that $\angle AMP = \angle CMQ$.

6. Consider a regular n-gon with n odd. Given two adjacent vertices A_1 and A_2, define the sequence (A_k) of vertices of the n-gon as follows: For $k \ge 3$, A_k is the vertex lying on the perpendicular bisector of $A_{k-2}A_{k-1}$. Find all n for which each vertex of the n-gon occurs in this sequence.

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