20-th Iberoamerican Mathematical Olympiad

September 27–28, 2005

First Day

1. Determine all triples (x,y,z) of real numbers such that

$$\begin{aligned} xyz &= 8 \\ x^2y+y^2z+z^2x &= 73 \\ x(y-z)^2+y(z-x)^2+z(x-y)^2 &= 98. \end{aligned}$$

2. A flea jumps in a straight line. It jumps first from point 0 to point 1. Afterwards, if its last jump was from a to b, then the next jump is from b to one of the points $b+(b-a)-1, b+(b-a), b+(b-a)+1$.

   Prove that if the flea arrived twice at the point n then it performed at least $\lceil 2\sqrt{n}\rceil$ jumps.

3. Let $p>3$ be a prime number such that

$$\sum_{i=1}^{p-1}\frac{1}{i^p}=\frac{n}{m},$$

   for some relatively prime integers m and n. Prove that n is divisible by p^3.

Second Day

4. Let a mod b denote the remainder of a when divided by b. Determine all pairs (a,p) of positive integers such that p is prime and

$$a \bmod p + a \bmod (2p) + a \bmod (3p) + a \bmod (4p) = a+p.$$

5. Let O be the circumcenter of the acute-angled triangle ABC, and let A_1 be a point on the smaller arc BC of the circumcircle of $\triangle ABC$. Denote by A_2 and A_3 on the sides ABC and AC, respectively, such that $\angle BA_1A_2=\angle OAC$ and $\angle CA_1A_3=\angle OAB$. Prove that the line A_2A_3 passes through the orthocenter of $\triangle ABC$.

6. Let n be a positive integer. The points $A_1, A_2, \dots, A_{2n}$ line on a line. Each of the points is colored in blue or red according to the following procedure: Draw n pairwise disjoint circles each with diameter A_iA_j for some pairs (i,j) with $i\neq j$ and such that every of the points $\{A_1,\dots,A_{2n}\}$ belongs to exactly one of the circles. Points on the same circle are colored by the same color.

   Determine the number of such colorings as we vary the circles and the distributions of colors.

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