# 19-th Iberoamerican Mathematical Olympiad

## Castellón, Spain, September 18–26, 2004

*First Day – September 21*

1. Some squares of a $1001 \times 1001$ board are to be colored according to the following rules:

   (i) If two squares share a side, then at least one of them must be colored;

   (ii) Among any six successive squares in a row or in a column some two adjacent ones must be both colored.

   Determine the smallest number of squares that need to be colored.

2. In the plane are given a circle with center $O$ and radius $r$ and a point $A$ outside the circle. For any point $M$ on the circle, let $N$ be the diametrically opposite point. Find the locus of the circumcenter of triangle $AMN$ when $M$ describes the circle.

3. Let $n$ and $k$ be positive integers either with $n$ odd or with both $n$ and $k$ even. Show that there exist integers $a$ and $b$ such that

$$\gcd(a,n) = \gcd(b,n) = 1 \quad \text{and} \quad k = a+b.$$

*Second Day – September 22*

4. Find all pairs $(a,b)$ of two-digit natural numbers such that both $100a+b$ and $201a+b$ are four-digit perfect squares.

5. In a scalene triangle $ABC$, points $A', B', C'$ are the intersection points of the internal bisectors of angles $A, B, C$ with the opposite sides, respectively. Let $BC$ meet the perpendicular bisector of $AA'$ at $A''$, $CA$ meet the perpendicular bisector of $BB'$ at $B''$, and $AB$ meet the perpendicular bisector of $CC'$ at $C''$. Prove that $A''$, $B''$ and $C''$ are collinear.

6. For a set $\mathscr{H}$ of points in a plane, we say that a point $P$ in the plane is an *intersection point* if there are distinct points $A, B, C, D$ in $\mathscr{H}$ such that the lines $AB$ and $CD$ intersect at $P$.
   Given a finite set $\mathscr{A}_0$ in the plane, sets $\mathscr{A}_1, \mathscr{A}_2, \dots$ are constructed inductively as follows: For every $j \geq 0$, $\mathscr{A}_{j+1}$ is the union of $\mathscr{A}_j$ and the set of intersection points of $\mathscr{A}_j$. Prove that if the union of all the sets $\mathscr{A}_j$ is finite, then $\mathscr{A}_j = \mathscr{A}_1$ for all $j \geq 1$.



The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com