18-th Iberoamerican Mathematical Olympiad

Mar del Plata, Argentina, September 2003

First Day – September 16

1. We are given two sequences of 2003 consecutive integers each, and a board with 2 rows and 2003 columns. Is it always possible to arrange the numbers from the first sequence in the first row and those from the second sequence in the second row of the board in such a way that the sums of entries by columns form a new sequence of 2003 consecutive integers?

2. Let C and D be points on the semicircle with diameter AB such that B and C are on different sides of AD. Denote by M,N,P the midpoints of AC,DB,CD, respectively. If O_A and O_B are the circumcenters of the triangles ACP and BDP, show that the lines O_AO_B and MN are parallel.

3. Pablo copied the following problem:

   - Consider all sequences $(x_0,x_1,\dots,x_{2003})$ of real numbers such that $x_0 = 1$, $0 \le x_1 \le 2x_0$, $0 \le x_2 \le 2x_1$, $\dots$, $0 \le x_{2003} \le 2x_{2002}$. Among all such sequences, find those that minimize the value of $S = \cdots$.

   Unfortunately, on the place where he wrote the expression for S the ink stained and all he could see was that S had the form

   $$S = \pm x_1 \pm x_2 \pm \cdots \pm x_{2002} + x_{2003}.$$

   Prove that Pablo can give a complete solution despite not knowing the complete formulation.

Second Day – September 17

4. Given the set $M = \{1,2,\dots,49\}$, find the maximum integer k for which there exists a k-element subset of M containing no six consecutive numbers. For this value of k, find the number of k-element subsets of M with the mentioned property.

5. In a square $ABCD$, P and Q are points on sides BC and CD respectively, distinct from their endpoints, such that $BP = CQ$. Let X and Y be arbitrary points on the segments AP and AQ, respectively. Show that there always exists a triangle with the sides congruent to BX, XY, and DY.

6. The sequences (a_n), (b_n) are defined by $a_0 = 1$, $b_0 = 4$ and

   $$a_{n+1} = a_n^{2001} + b_n, \quad b_{n+1} = b_n^{2001} + a_n \quad \text{for } n \ge 0.$$

   Prove that no term of any of these sequences is a multiple of 2003.

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