16-th Iberoamerican Mathematical Olympiad

Minas, Uruguay, September 24–29, 2001

First Day

1. We call a natural number n *coarse* if it has the following properties:

   (i) All the digits of n are greater than 1;

   (ii) The product of four of its digits always divides n.

   Show that for each positive integer k there exists a coarse number with k digits.

2. The incircle of a triangle ABC is centered at O and tangent to BC, CA, AB at points X, Y, Z, respectively. The lines BO and CO meet the line YZ at P and Q. Prove that if X is equidistant from P and Q, then $\triangle ABC$ is isosceles.

3. Let $S_1, S_2, \dots, S_k$ $(k \geq 2)$ be subsets of an n-element set S such that each S_i has at least r elements. Prove that there exist i and j with $1 \leq i < j \leq k$ such that

$$|S_i \cap S_j| \geq r - \frac{nk}{4k-4}.$$

Second Day

4. Determine the largest possible number of increasing arithmetic progressions with three terms contained in a sequence $a_1 < a_2 < \dots < a_n$ of $n \geq 3$ real numbers.

5. The squares of a 2000×2001 board are assigned integer coordinates (x, y) with $0 \leq x \leq 1999$ and $0 \leq y \leq 2000$. A ship is moved on the board as follows. Assume that, prior to a move, the ship have position (x, y) and velocity (h, v), where x, y, h, v are integers. Then the move consists of changing its velocity to (h', v'), where $h' - h, v' - v \in \{-1, 0, 1\}$, and its position to (x', y'), where x' and y' are the remainders of $x + h'$ modulo 2000 and of $y + v'$ modulo 2001, respectively.

   There are two ships on the board: a Martian ship and a Human trying to capture it. Each ship is initially positioned at a square of the board and has the velocity $(0, 0)$. The Human makes the first move; thereafter, the ships are alternately moved. Is there a strategy for the Human to capture the Martian, independent of the initial positions and the Martian's moves?

6. Prove that it is not possible to cover a unit square by five congruent squares with side smaller than $1/2$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com