43-rd International Mathematical Olympiad

Glasgow, United Kingdom, July 19–30, 2002

First Day – July 24

1. Let n be a positive integer. Each point (x,y) in the plane, where x and y are nonnegative integers with $x+y=n$, is colored red or blue, subject to the following condition: If a point (x,y) is red, then so are all points (x',y') with $x' \le x$ and $y' \le y$. Let A be the number of ways to choose n blue points with distinct x-coordinates, and let B be the number of ways to choose n blue points with distinct y-coordinates. Prove that $A = B$.

   (Colombia)

2. The circle S has center O, and BC is a diameter of S. Let A be a point of S such that $\measuredangle AOB < 120^\circ$. Let D be the midpoint of the arc AB that does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and at F. Prove that I is the incenter of the triangle CEF.

   (South Korea)

3. Find all pairs of positive integers $m,n \ge 3$ for which there exist infinitely many positive integers a such that
$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$
is itself an integer.

   (Romania)

Second Day – July 25

4. Let $n \ge 2$ be a positive integer, with divisors $1 = d_1 < d_2 < \cdots < d_k = n$. Prove that $d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k$ is always less than n^2, and determine when it is a divisor of n^2.

   (Romania)

5. Find all functions f from the reals to the reals such that
$$(f(x)+f(z))(f(y)+f(t)) = f(xy-zt) + f(xt+yz)$$
for all real x,y,z,t.

   (India)

6. Let $n \ge 3$ be a positive integer. Let $C_1, C_2, C_3, \ldots, C_n$ be unit circles in the plane, with centers $O_1, O_2, O_3, \ldots, O_n$ respectively. If no line meets more than two of the circles, prove that
$$\sum_{1 \le i < j \le n} \frac{1}{O_iO_j} \le \frac{(n-1)\pi}{4}.$$

   (Ukraine)

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