38-th International Mathematical Olympiad

Mar del Plata, Argentina, July 18–31, 1997

First Day – July 24

1. An infinite square grid is colored in the chessboard pattern. For any pair of positive integers m,n consider a right-angled triangle whose vertices are grid points and whose legs, of lengths m and n, run along the lines of the grid. Let S_b be the total area of the black part of the triangle and S_w the total area of its white part. Define the function $f(m,n) = |S_b - S_w|$.

   (a) Calculate $f(m,n)$ for all m,n that have the same parity.

   (b) Prove that $f(m,n) \leq \frac{1}{2}\max(m,n)$.

   (c) Show that $f(m,n)$ is not bounded from above. *(Belarus)*

2. In triangle ABC the angle at A is the smallest. A line through A meets the circumcircle again at the point U lying on the arc BC opposite to A. The perpendicular bisectors of CA and AB meet AU at V and W, respectively, and the lines CV, BW meet at T. Show that $AU = TB + TC$. *(Great Britain)*

3. Let $x_1, x_2, \dots, x_n$ be real numbers satisfying the conditions $|x_1 + x_2 + \cdots + x_n| = 1$ and $|x_i| \leq \frac{n+1}{2}$ for $i = 1, 2, \dots, n$. Show that there exists a permutation $y_1, \dots, y_n$ of the sequence $x_1, \dots, x_n$ such that
$$|y_1 + 2y_2 + \cdots + ny_n| \leq \frac{n+1}{2}.$$
(Russia)

Second Day – July 25

4. An $n \times n$ matrix with entries from $\{1, 2, \dots, 2n-1\}$ is called a *silver matrix* if for each i the union of the ith row and the ith column contains $2n-1$ distinct entries. Show that:

   (a) There exist no silver matrices for $n = 1997$.

   (b) Silver matrices exist for infinitely many values of n. *(Iran)*

5. Find all pairs of integers $x, y \geq 1$ satisfying the equation $x^{y^2} = y^x$. *(Czech Republic)*

6. For a positive integer n, let $f(n)$ denote the number of ways to represent n as the sum of powers of 2 with nonnegative integer exponents. Representations that differ only in the ordering in their summands are not considered to be distinct. (For instance, $f(4) = 4$ because the number 4 can be represented in the following four ways: 4; 2+2; 2+1+1; 1+1+1+1.) Prove the inequality
$$2^{n^2/4} < f(2^n) < 2^{n^2/2} \quad \text{for all } n \geq 3.$$
(Lithuania)

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