13-th International Mathematical Olympiad

Bratislava – Žilina, Czechoslovakia, July 10–21, 1971

First Day – July 13

1. Prove that the following statement is true for $n = 3$ and for $n = 5$, and false for all other $n > 2$:
   For any real numbers $a_1, a_2, \dots, a_n$,
   $$(a_1 - a_2)(a_1 - a_3)\cdots(a_1 - a_n) + (a_2 - a_1)(a_2 - a_3)\cdots(a_2 - a_n) + \dots + (a_n - a_1)(a_n - a_2)\cdots(a_n - a_{n-1}) \geq 0.$$
   (Hungary)

2. Given a convex polyhedron P_1 with 9 vertices $A_1, \dots, A_9$, let us denote by $P_2, P_3, \dots, P_9$ the images of P_1 under the translations mapping the vertex A_1 to $A_2, A_3, \dots, A_9$, respectively. Prove that among the polyhedra $P_1, \dots, P_9$ at least two have a common interior point. *(Soviet Union)*

3. Prove that the sequence $2^n - 3$ $(n > 1)$ contains a subsequence of numbers relatively prime in pairs. *(Poland)*

Second Day – July 14

4. Given a tetrahedron $ABCD$ all of whose faces are acute-angled triangles, set $\sigma = \measuredangle DAB + \measuredangle BCD - \measuredangle ABC - \measuredangle CDA$.

   Consider all closed broken lines $XYZTX$ whose vertices X, Y, Z, T lie in the interior of segments AB, BC, CD, DA respectively. Prove that:

   (a) if $\sigma \neq 0$, then there is no broken line $XYZT$ of minimal length;

   (b) if $\sigma = 0$, then there are infinitely many such broken lines of minimal length. That length equals $2AC\sin(\alpha/2)$, where
   $$\alpha = \measuredangle BAC + \measuredangle CAD + \measuredangle DAB.$$
   (Netherlands)

5. Prove that for every natural number $m \geq 1$ there exists a finite set S_m of points in the plane satisfying the following condition: If A is any point in S_m, then there are exactly m points in S_m whose distance to A equals 1. *(Bulgaria)*

6. Consider the $n \times n$ array of nonnegative integers
   $$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},$$
   with the following property: If an element a_{ij} is zero, then the sum of the elements of the ith row and the jth column is greater than or equal to n. Prove that the sum of all the elements is greater than or equal to $\geq \frac{1}{2}n^2$. *(Sweden)*

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