10-th International Mathematical Olympiad

Moscow – Leningrad, Soviet Union, July 5–18, 1968

First Day

1. Prove that there exists a unique triangle whose side lengths are consecutive natural numbers and one of whose angles is twice the measure of one of the others. *(Romania)*

2. Find all positive integers x for which $p(x) = x^2 - 10x - 22$, where $p(x)$ denotes the product of the digits of x. *(Czechoslovakia)*

3. Let a, b, c be real numbers. Prove that the system of equations

$$\begin{cases} ax_1^2 + bx_1 + c = x_2, \\ ax_2^2 + bx_2 + c = x_3, \\ \cdots\cdots\cdots\cdots \\ ax_{n-1}^2 + bx_{n-1} + c = x_n, \\ ax_n^2 + bx_n + c = x_1, \end{cases}$$

   (a) has no real solutions if $(b-1)^2 - 4ac < 0$;

   (b) has a unique real solution if $(b-1)^2 - 4ac = 0$;

   (c) has more than one real solution if $(b-1)^2 - 4ac > 0$. *(Bulgaria)*

Second Day

4. Prove that in any tetrahedron there is a vertex such that the lengths of its sides through that vertex are sides of a triangle. *(Poland)*

5. Let $a > 0$ be a real number and $f(x)$ a real function defined on all of $\mathbb{R}$, satisfying for all $x \in \mathbb{R}$,

$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - f(x)^2}.$$

   (a) Prove that the function f is periodic; i.e., there exists $b > 0$ such that for all x, $f(x+b) = f(x)$.

   (b) Give an example of such a nonconstant function for $a = 1$. *(DR Germany)*

6. Let $[x]$ denote the integer part of x, i.e., the greatest integer not exceeding x. If n is a positive integer, express as a simple function of n the sum

$$\left[\frac{n+1}{2}\right] + \left[\frac{n+2}{4}\right] + \cdots + \left[\frac{n+2^i}{2^{i+1}}\right] + \cdots.$$ *(Great Britain)*

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