6-th International Mathematical Olympiad

Moscow, Soviet Union, June 30–July 10, 1964

First Day

1. (a) Find all natural numbers n such that the number $2^n - 1$ is divisible by 7.

   (b) Prove that for all natural numbers n the number $2^n + 1$ is not divisible by 7.

   (Czechoslovakia)

2. Denote by a, b, c the lengths of the sides of a triangle. Prove that

$$a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) \leq 3abc.$$

   (Hungary)

3. The incircle is inscribed in a triangle ABC with sides a, b, c. Three tangents to the incircle are drawn, each of which is parallel to one side of the triangle ABC. These tangents form three smaller triangles (internal to $\triangle ABC$) with the sides of $\triangle ABC$. In each of these triangles an incircle is inscribed. Determine the sum of areas of all four incircles. *(Yugoslavia)*

Second Day

4. Each of 17 students talked with every other student. They all talked about three different topics. Each pair of students talked about one topic. Prove that there are three students that talked about the same topic among themselves. *(Hungary)*

5. Five points are given in the plane. Among the lines that connect these five points, no two coincide and no two are parallel or perpendicular. Through each point we construct an altitude to each of the other lines. What is the maximal number of intersection points of these altitudes (excluding the initial five points)? *(Romania)*

6. Given a tetrahedron $ABCD$, let D_1 be the centroid of the triangle ABC and let A_1, B_1, C_1 be the intersection points of the lines parallel to DD_1 and passing through the points A, B, C with the opposite faces of the tetrahedron. Prove that the volume of the tetrahedron $ABCD$ is one-third the volume of the tetrahedron $A_1B_1C_1D_1$. Does the result remain true if the point D_1 is replaced with any point inside the triangle ABC?

   (Poland)

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