Hungarian Mathematical Olympiad 1997/98

Final Round

Grades 11 and 12

1. Prove that the arithmetic mean of the numbers $2\sin 2^\circ$, $4\sin 4^\circ$, $6\sin 6^\circ$, $\dots, 180\sin 180^\circ$ is equal to $\cot 1^\circ$.

2. Let P, Q, R be points on the sides AB, BC, CA of a triangle ABC, respectively, and let A', B', C' be points on PR, QP, RQ, respectively, such that $A'B' \parallel AB$, $B'C' \parallel BC$, $C'A' \parallel CA$. Show that $\dfrac{AB}{A'B'} = \dfrac{S_{PQR}}{S_{A'B'C'}}$.

3. Find all positive integers n for which there exist positive integers x, y such that $x \geq y$, $\operatorname{lcm}(x, y) = n!$, and $\gcd(x, y) = 1998$. For which n is the number of pairs (x, y) smaller than 1998?

Grades 11 and 12 – specialized math classes

1. Let $a_1, a_2, \cdots, a_r$ be the numbers among $1, 2, \dots, 1998$ that are coprime to 1998. For which k do the numbers ka_i, $i = 1, 2, \dots, 1998$, give different remainders upon division by 1998?

2. In a small town T there are several clubs, denoted by C_i, all with the same number of members, such that for some real number p:

   (i) $p|T| = |C_i|$ for all i;

   (ii) For any number of clubs $C, C_1, \dots, C_r$, we have $p|C_1 \cap \cdots \cap C_r| = |C \cap C_1 \cap \cdots \cap C_r|$.

   We know that there were k clubs in 1996, $k+1$ in 1997, and $k+2$ in 1998. The above conditions were satisfied in these years and the number of citizens in T did not change, but the number of citizens attending any club increased by 3240 in 1996-97 and by 2916 in 1997-98.

   All these statements are true for another, larger town. By how many people is this town larger than T?

3. In a triangle ABC, points P and Q are given on the side AB such that the inradii of the triangles APC and QBC are equal. Prove that the inradii of the triangles AQC and PBC are also equal.

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