Hungarian Mathematical Olympiad 2005/06

Final Round

High Schools

1. For each nonnegative integer n, define $t(0) = t(1) = 0$, $t(2) = 1$ and for $n > 2$ define $f(n)$ to be the smallest natural number not dividing n. Let $T(n) = t(t(t(n)))$. Evaluate

$$S = T(1) + T(2) + \cdots + T(2006).$$

2. Let A and B be two vertices of a tree with 2006 edges. We move along the edges starting from A and would like to get to N without turning back. At any vertex we choose the next edge among all possible edges (i.e. excluding the one on which we arrived) with equal probability. Over all possible trees and choices of vertices A and B, find the minimum probability of getting from A to B.

3. A unit circle k with center K and a line e are given in the plane so that $KO = 2$, where O is the projection of K on e. Consider the set H of all circles centeerd on e which are externally tangent to k. Show that there are a point P in the plane and an angle $\alpha > 0$ such that for any circle in H with diameter AB on e it holds that $\angle APB = \alpha$. Determine α and the location of P.

Specialized Math Schools

1. For every non-rectangular triangle $A_1B_1C_1$ with the altitudes A_1A_2, B_1B_2, C_1C_2, we call $A_2B_2C_2$ the *pedal triangle* of $A_1B_1C_1$. Construct a sequence of triangles $(A_kB_kC_k)$ such that $A_{i+1}B_{i+1}C_{i+1}$ is the pedal triangle of $AA_iB_iC_i$. How many pairwise non-similar triangles $A_1B_1C_1$ with integer angles (in degrees) are there for which there is a $k > 1$ such that $\triangle A_kB_kC_k \sim \triangle A_1B_1C_1$?

2. Denote by $d(n)$ the number of positive divisors of n. Suppose that r and s are positive integers with the property that $d(ks) \geq d(kr)$ for each $k \in \mathbb{N}$. Prove that r divides s.

3. A cube of edge n is made of n^3 unit cubes so that there are $6n^2$ unit squares on the surface of the cube. At most how many of these squares can be chosen so that no two of them have a common edge?

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