Hungarian Mathematical Olympiad 2002/03

Final Round

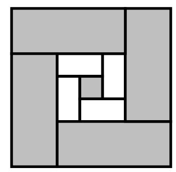
Category 1

1. Solve the equation $(x^2 + 5x + 1)(x^2 + 4x) = 20(x + 1)^2$ in $\mathbb{R}$.
2. The medians s_a, s_b, s_c and the circumradius R of a triangle satisfy

$$6R^2 = s_a^2 + s_b^2 + s_c^2,$$

prove that the triangle is right-angled.

3. In the first step we draw a unit square and shade it. In the second step we draw four unshaded rectangles around the square, with one side being twice the other. In the third step we draw shaded rectangles around the newly obtained square, again with one side being twice the other (see the figure). We continue drawing alternately shaded and unshaded rectangles around the obtained squares. After the n-th step, find the absolute value of the difference between the shaded and unshaded areas in terms of n.



Category 2

1. The excircles of triangle ABC touch the sides AB, BC, CA at points C', A', B' respectively. The midpoints of the altitudes from A, B and C are X, Y, Z, respectively. Prove that lines XA', YB' and ZC' pass through the incenter.
2. Let $n \geq 2$ be an integer. Denote by a_n the greatest n-digit natural number which is neither a difference nor a sum of two squares.
   (a) Find a_n as a function of n.
   (b) Find the least n for which the sum of the squares of the digits of a_n is a square.
3. Three faces of a regular tetrahedron $ABCD$ are white, the one opposite D is black, and the tetrahedron lies on plane S with the black face down. The tetrahedron can be rolled on the plane by rotation about its edges. If after some rolling the tetrahedron returns to the original place, show that the face on plane S cannot be white.

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Category 3

1. A set H of points in the plane is called *nice* if any three-element subset of H has a symmetry axis. Prove the following statements:

   (a) A nice set is not necessarily symmetrical.

   (b) If a nice set H has exactly 2003 points, then all of them must lie on a line.

2. We color the vertices of a 2003-gon with red, blue, and green such that adjacent vertices cannot have the same color. In how many ways can this be done?

3. For a fixed positive integer t, let $f_t(n)$ denote the number of integers k such that $1 \leq k \leq n$ and $\binom{k}{t}$ is odd. Prove that if n is a sufficiently large power of 2, then $\dfrac{f_t(n)}{n} = \dfrac{1}{2^r}$, where r is an integer depending only on t.

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