7-th Hong Kong (China) Mathematical Olympiad 2004

December 4, 2004

1. Let $a_1, a_2, \dots, a_{n+1}$ $(n \geq 2)$ be positive numbers with $a_2 - a_1 = a_3 - a_2 = \cdots = a_{n+1} - a_n$. Prove that

$$\frac{1}{a_2^2} + \frac{1}{a_3^2} + \cdots + \frac{1}{a_n^2} \leq \frac{n-1}{2} \cdot \frac{a_1 a_n + a_2 a_{n+1}}{a_1 a_2 a_n a_{n+1}}.$$

2. In a school there are b teachers and c students. Suppose that

   (i) each teacher teaches exactly k students, and

   (ii) for any two (distinct) students, exactly h teachers teach both of them.

   Prove that $\frac{b}{h} = \frac{c(c-1)}{k(k-1)}$.

3. Points P and Q are taken on the sides AB and AC of a triangle ABC respectively such that $\angle APC = \angle AQB = 45^\circ$. The line through P perpendicular to AB intersects BQ at S, and the line through Q perpendicular to AC intersects CP at R. Let D be the foot of the altitude of $\triangle ABC$ from A. Prove that SR and BC are parallel and that PS, AD, QR are concurrent.

4. Let $S = \{1, 2, \dots, 100\}$. Find the number of functions $f : S \to S$ satisfying the following conditions:

   (i) $f(1) = 1$;

   (ii) f is bijective;

   (iii) $f(n) = f(g(n)) f(h(n))$ for all $n \in S$, where $g(n)$ and $h(n)$ are the positive integers with $g(n) \leq h(n)$ and $g(n)h(n) = n$ that minimize $h(n) - g(n)$. (For instance, $g(80) = 8$, $h(80) = 10$.)

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