10-th Hungary–Israel Binational Mathematical Competition 1999

First Day

1. Let $f(x)$ be a polynomial of degree at least 2. The sequence $g_n(x)$ is defined by $g_1(x) = f(x)$ and $g_{n+1}(x) = f(g_n(x))$ for $n \in \mathbb{N}$. Let r_n be the average of the roots of $g_n(x)$. If $r_{19} = 99$, determine r_{99}.

2. In a plane are given $2n+1$ lines, no two of which are parallel and no three concurrent. Every three of them form a non-right triangle. Among all such triangles, find the maximum possible number of acute-angled ones.

3. Find all functions $f : \mathbb{Q} \to \mathbb{R}$ such that for all $x, y \in \mathbb{Q}$,
$$f(x+y) = f(x)f(y) - f(xy) + 1.$$

Second Day

4. For a positive integer c, define the sequence (a_n) by
$$a_1 = c, \quad a_{n+1} = ca_n + \sqrt{(c^2-1)(a_n^2-1)} \text{ for } n = 1, 2, \dots$$
Prove that all a_n are positive integers.

5. Define $f(x,y,z) = \dfrac{x^2+y^2+z^2}{x+y+z}$ for all real x,y,z with $x+y+z \neq 0$. Find a point (x_0, y_0, z_0) such that $0 < x_0^2 + y_0^2 + z_0^2 < \frac{1}{1999}$ and $1.999 < f(x_0, y_0, z_0) < 2$.

6. An exam consists of four multiple choice questions, each question having three choices. A group of examinees took the exam. It turned out that for any three examinees there was at least one question to which they gave three different answers. What is the maximum possible number of examinees in this group?

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