9-th Hungary–Israel Binational Mathematical Competition 1998

Haifa, Israel, April 2–10

Individual competition

First Day

1. A player is playing the following game. In each turn he flips a coin and guesses the outcome. If his guess is correct, he gains 1 point; otherwise he loses all his points. Initially the player has no points, and plays the game until he has 2 points.

   (a) Find the probability p_n that the game ends after exactly n flips.

   (b) What is the expected number of flips needed to finish the game?

2. A triangle ABC is inscribed in a circle with center O and radius R. If the inradii of the triangles OBC, OCA, OAB are r_1, r_2, r_3, respectively, prove that
$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} \geq \frac{4\sqrt{3}+6}{R}.$$

3. Let a, b, c, m, n be positive integers. Consider the trinomial $f(x) = ax^2 + bx + c$. Show that there exist n consecutive natural numbers $a_1, a_2, \dots, a_n$ such that each of the numbers $f(a_1), f(a_2), \dots, f(a_n)$ has at least m different prime factors.

Second Day

4. Find all positive integers x and y such that $5^x - 3^y = 16$.

5. On the sides of a convex hexagon $ABCDEF$, equilateral triangles are constructd in its exterior. Prove that the third vertices of these six triangles are vertices of a regular hexagon if and only if the initial hexagon is *affine regular*. (A hexagon is called affine regular if it is centrally symmetric and any two opposite sides are parallel to the diagonal determine by the remaining two vertices.)

6. Let n be a positive integer. We consider the set Π of all partitions of n into a sum of positive integers (the order is irrelevant). For every partition α, let $a_k(\alpha)$ be the number of summands in α that are equal to k, $k = 1, 2, \dots, n$. Prove that
$$\sum_{\alpha\in\Pi} \frac{1}{1^{a_1(\alpha)}a_1(\alpha)! \cdot 2^{a_2(\alpha)}a_2(\alpha)! \cdots n^{a_n(\alpha)}a_n(\alpha)!} = 1.$$

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