4-th Hungary–Israel Binational Mathematical Competition 1993

Budapest, Hungary

Individual competition – April 21

1. Find all pairs of coprime natural numbers a and b such that the fraction a/b is written in the decimal system as $b.a$.

2. Determine all polynomials $f(x)$ with real coefficients that satisfy
$$f(x^2 - 2x) = f(x-2)^2 \quad \text{for all } x.$$

3. Distinct points A, B, C, D, E are given in this order on a semicircle with radius 1. Prove that
$$AB^2 + BC^2 + CD^2 + DE^2 + AB \cdot BC \cdot CD + BC \cdot CD \cdot DE < 4.$$

4. Find the largest possible number of rooks that can be placed on a $3n \times 3n$ chessboard so that each rook is attacked by at most one rook.

Team competition – April 22

In the questions below: G is a finite group; $H \leq G$ a subgroup of G; $|G : H|$ the index of H in G; $|X|$ the number of elements of $X \subset G$; $Z(G)$ the center of G; G' the commutator subgroup of G; $N_G(H)$ the normalizer of H in G; $C_G(H)$ the centralizer of H in G; and S_n the n-th symmetric group.

1. Suppose $k \geq 2$ is an integer such that for all $x, y \in G$ and $i \in \{k-1, k, k+1\}$ the relation $(xy)^i = x^i y^i$ holds. Show that G is Abelian.

2. Suppose that $n \geq 1$ is such that the mapping $x \mapsto x^n$ from G to itself is an isomorphism. Prove that for each $a \in G$, $a^{n-1} \in Z(G)$.

3. Show that every element of S_n is a product of 2-cycles.

4. Let $H \leq G$ and $a, b \in G$. Prove that $|aH \cap Hb|$ is either zero or a divisor of $|H|$.

5. Let $H \leq G$, $|H| = 3$. What can be said about $|N_G(H) : C_G(H)|$?

6. Let $a, b \in G$. Suppose that $ab^2 = b^3 a$ and $ba^2 = a^3 b$. Prove that $a = b = 1$.

7. Assume $|G'| = 2$. Prove that $|G : G'|$ is even.