18-th Hungary–Israel Binational Mathematical Competition 2007

First Day – March 27

1. Your task is to organize a fair procedure to randomly select one of n people, each one having the same probability $\frac{1}{n}$ to be selected. For this purpose, you are allowed to choose two real numbers $p_1, p_2 \in (0,1)$ and order two coins whose probabilities of tossing heads equal p_1 and p_2 respectively. Before starting the procedure you are also supposed to announce an upper bound on the total number of flipping the two coins. Describe the procedure that achieves this goal under the given conditions.

2. Let a,b,c,d be real numbers such that $a^2 \le 1$, $a_2 + b^2 \le 5$, $a^2 + b^2 + c^2 \le 14$, and $a^2 + b^2 + c^2 + d^2 \le 30$. Prove that $a + b + c + d \le 10$.

3. Let AB be a diameter of a circle of unit radius, and let P be a fixed point on AB. If a variable line through P meets the circle at C and D, find the maximum possible area of the quadrilateral with vertices at A,B,C,D.

Second Day – March 28

4. A given rectangle R is divided into $n \times m$ small rectangles (not necessarily congruent) by straight lines parallel to its sides. At least how many appropriately selected rectangles areas should be known in order to determine the area of R?

5. Given an ellipse e in the plane, find the locus of the points P in space such that the cone of apex P and directrix e is a right circular cone.

6. Let $t \ge 3$ be a given real number and assume that the polynomial $f(x)$ satisfies

$$|f(k) - t^k| < 1 \quad \text{for } k = 0,1,2,\ldots,n.$$

Prove that the degree of f is at least n.

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