16-th Hungary–Israel Binational Mathematical Competition 2005

First Day – Budapest, April 18

1. Squares ABB_1A_2 and BCC_1B_2 are externally drawn on the hypotenuse AB and on the leg BC of a right triangle ABC. Show that the lines CA_2 and AB_2 meet on the perimeter of a square with the vertices on the perimeter of triangle ABC.

2. Let f be an increasing mapping from the family of subsets of a given finite set H into itself, i.e. such that for every $X \subseteq Y \subseteq H$ we have $f(X) \subseteq f(Y) \subseteq H$. Prove that there exists a subset H_0 of H such that $f(H_0) = H_0$.

3. Find all sequences $x_1, \dots, x_n$ of distinct positive integers such that

$$\frac{1}{2} = \frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_n^2}.$$

Second Day – Budapest, April 19

4. Does there exist a sequence of 2005 consecutive positive integers that contains exactly 25 prime numbers?

5. Let F_n be the n-th Fibonacci number (where $F_1 = F_2 = 1$). Consider the functions

$$\begin{aligned} f_n(x) &= ||\dots|||x| - F_n| - F_{n-1}| - \cdots - F_2| - F_1|, \\ g_n(x) &= |\dots||x-1|-1| - \cdots - 1| \qquad (F_1 + \cdots + F_n \text{ one's}). \end{aligned}$$

   Show that $f_n(x) = g_n(x)$ for every real number x.

6. There are seven rods erected at the vertices of a regular heptagonal area. The top of each rod is connected to the top of its second neighbor by a straight piece of wire so that, looking from above, one sees each wire crossing exactly two others. Is it possible to set the respective heights of the rods in such a way that no four tops of the rods are coplanar and each wire passes one of the crossings from above and the other one from below?

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com