14-th Hungary–Israel Binational Mathematical Competition 2003

First Day – Budapest, April 9

1. If $x_1, x_2, \dots, x_n$ are positive numbers, prove the inequality

$$\frac{x_1^3}{x_1^2 + x_1x_2 + x_2^2} + \frac{x_2^3}{x_2^2 + x_2x_3 + x_3^2} + \dots + \frac{x_n^3}{x_n^2 + x_nx_1 + x_1^2}$$
$$\geq \frac{x_1 + x_2 + \dots + x_n}{3}.$$

2. Let ABC be an acute-angled triangle. The tangents to its circumcircle at A, B, C form a triangle PQR with $C \in PQ$ and $B \in PR$. Let C_1 be the foot of the altitude from C in $\triangle ABC$. Prove that CC_1 bisects $\angle QC_1P$.

3. Let $d > 0$ be an arbitrary real number. Consider the set

$$S_n(d) = \left\{ s = \frac{1}{x_1} + \dots + \frac{1}{x_n} \mid x_i \in \mathbb{N},\ s < d \right\}.$$

   Prove that $S_n(d)$ has a maximum element.

Second Day – Budapest, April 10

4. Two players play the following game. They alternately write divisors of 100! on the blackboard, not repeating any of the numbers written before. The player after whose move the greatest common divisor of the written numbers equals 1, loses the game. Which player has a winning strategy?

5. Let M be a point inside a triangle ABC. The lines AM, BM, CM intersect BC, CA, AB at A_1, B_1, C_1, respectively. Assume that

$$S_{MAC_1} + S_{MBA_1} + S_{MCB_1} = S_{MA_1C} + S_{MB_1A} + S_{MC_1B}.$$

   Prove that one of the lines AA_1, BB_1, CC_1 is a median of the triangle ABC.

6. Let n be a positive integer. Show that there exist three distinct integers between n^2 and $n^2 + n + 3\sqrt{n}$, such that one of them divides the product of the other two.

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