25-th Hellenic Mathematical Olympiad

Athens, February 23, 2008

Juniors

1. Let p, q denote distinct prime numbers and k, l positive integers. Find all positive divisors of the numbers: (a) $A = p^k$; (b) $B = p^k q^l$; (c) $C = 1944$.

2. If x, y, z are positive real numbers with $x^2 + y^2 + z^2 = 3$, prove that
$$\frac{3}{2} < \frac{1+y^2}{x+2} + \frac{1+z^2}{y+2} + \frac{1+x^2}{z+2} < 3.$$

3. Find the greatest positive integer x for which $A = 2^{182} + 4^x + 8^{700}$ is a perfect square.

4. Let $ABCD$ be a trapezoid with $AD = a$, $AB = 2a$, $BC = 3a$ and $\angle A = \angle B = 90^\circ$. Let E and Z be the midpoints of AB and CD respectively, and let I be the foot of the perpendicular from Z to BC. Prove that
   (a) triangle BDC is isosceles;
   (b) the midpoint O of EZ is the barycenter of $\triangle BDZ$;
   (c) the lines AZ and DI intersect on the line BO.

Seniors

1. A computer generates all pairs of real numbers $x, y \in (0,1)$ for which the numbers $a = x + my$ and $b = y + mx$ are both integers, where m is a given positive integer. Finding one such pair (x, y) takes 5 seconds. Find m is the computer needs 595 seconds to find all possible ordered pairs (x, y).

2. Find all integers x and prime numbers p satisfying $x^8 + 2^{2^x+2} = p$.

3. A triangle ABC with orthocenter H is inscribed in a circle with center K and radius 1, where the angles at B and C are non-obtuse. If the lines HK and BC meet at point S such that $SK(SK - SH) = 1$, compute the area of the concave quadrilateral $ABHC$.

4. If $a_1, a_2, \ldots, a_n$ are positive integers and $k = \max\{a_1, \ldots, a_n\}$, $t = \min\{a_1, \ldots, a_n\}$, prove the inequality
$$\left(\frac{a_1^2 + \cdots + a_n^2}{a_1 + \cdots + a_n}\right)^{\frac{kn}{t}} \geq a_1 a_2 \cdots a_n.$$
   When does equality hold?

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com