24-th Hellenic Mathematical Olympiad

February 24, 2007

Juniors

1. In a triangle ABC with the incenter I, the angle bisector AD meets the circumcircle of triangle BIC at point $N \neq I$.

   (a) Express the angles of $\triangle BCN$ in terms of the angles of $\triangle ABC$.

   (b) Show that the circumcenter of triangle BIC is at the intersection of AI and the circumcenter of ABC.

2. If n is is an integer such that $4n+3$ is divisible by 11, find the from of n and the remainder of n^4 upon division by 11.

3. For an integer n, denote $A = \sqrt{n^2+24}$ and $B = \sqrt{n^2-9}$. Find all values of n for which $A - B$ is an integer.

4. Each of the 50 students in a class sent greeting cards to 25 of the others. Prove that there exist two students who greeted each other.

Seniors

1. Find all natural numbers n for which the number $2007+n^4$ is a perfect square.

2. If a, b, c are the sides of a triangle, prove that

$$\frac{(c+a-b)^4}{a(a+b-c)} + \frac{(a+b-c)^4}{b(b+c-a)} + \frac{(b+c-a)^4}{c(c+a-b)} \geq ab+bc+ca.$$

3. In a circular ring with radii $11r$ and $9r$ we put circles of radius r which are tangent to the boundary circles and do not overlap. Determine the maximum number of circles that can be put this way. (You may use that $9.94 < \sqrt{99} < 9.95$.)

4. Given a 2007×2007 array of numbers 1 and -1, let A_i denote the product of the entries in the i-th row and B_j denote the product of the entries in the j-th column. Show that

$$A_1 + A_2 + \cdots + A_{2007} + B_1 + B_2 + \cdots + B_{2007} \neq 0.$$