23-rd Hellenic Mathematical Olympiad 2006

February 25, 2006

Juniors

1. Let P be an interior point of an equilateral triangle ABC. Show that there is a triangle with sides congruent to PA, PB, PC.

2. Find all pairs of positive integers (x, y) such that $2x^y - y = 2005$.

3. Prove that among any 27 distinct positive integers less than 100 there exist two that are not coprime.

4. If real numbers x and y satisfy the condition $x^2 + xy + y^2 = 1$, find the minimum and maximum value of $K = x^3y + xy^3$.

Seniors

1. Determine the number of five-digit natural numbers whose digits form a non-decreasing sequence.

2. Prove that if n is a positive integer, then the equation
$$x + y + \frac{1}{x} + \frac{1}{y} = 3n$$
has no solution in rational numbers x, y.

3. Let L, M, N be points on the sides BC, CA, AB respectively such that AL bisects the angle A and AL, BN and CM meet at a point. Prove that if $\angle ALB = \angle ANM$ then $\angle MNL = 90^\circ$.

4. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the conditions
   (i) $f(x + y + z) \leq 3(xy + yz + zx)$ for all real x, y, z, and
   (ii) there is a function $g : \mathbb{R} \to \mathbb{R}$ and a natural number n such that $g(g(x)) = x^{2n+1}$ and $f(g(x)) = g(x)^2$ for every real x?

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com