22-nd Hellenic Mathematical Olympiad 2005

February 12, 2005

Juniors

1. We are given a trapezoid with $AB \parallel CD$, $CD = 2AB$ and $DB \perp BC$. Let E be the intersection of lines DA and CB, and M be the midpoint of DC.
   (a) Prove that $ABMD$ is a rhombus.
   (b) Prove that triangle CDE is isosceles.
   (c) If AM and BD meet at O, and OE and AB meet at N, prove that line DN bisects segment EB.

2. If $f(n) = \dfrac{2n+1+\sqrt{n(n+1)}}{\sqrt{n+1}+\sqrt{n}}$ for all positive integers n, evaluate
   (a) $f(1)$,
   (b) the sum $A = f(1) + f(2) + \cdots + f(400)$.

3. Let A be a given point outside a given circle. Determine points B, C, D on the circle such that the quadrilateral $ABCD$ is convex and has the maximum area.

4. Find all nonzero integers a, b, c, d with $a > b > c > d$ that satisfy
$$ab + cd = 34 \quad \text{and} \quad ac - bd = 19.$$

Seniors

1. Determine all polynomials $P(x)$ with real coefficients such that $P(2) = 12$ and
$$P(x^2) = x^2(x^2+1)P(x) \quad \text{for all } x \in \mathbb{R}.$$

2. The sequence (a_n) is defined by $a_1 = 1$ and $a_n = a_{n-1} + \dfrac{1}{n^3}$ for $n > 1$.
   (a) Prove that $a_n < \dfrac{5}{4}$ for all n.
   (b) Given $\varepsilon > 0$, find the smallest natural number n_0 such that $|a_{n+1} - a_n| < \varepsilon$ for all $n > n_0$.

3. Let k be a given positive integer. If (x_0, y_0) is a solution to the equation
$$x^3 + y^3 - 2y(x^2 - xy + y^2) = k^2(x - y)$$
in distinct nonzero integers, show that:

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(a) The equation has finitely many solutions in distinct integers (x, y);

(b) There are at least 11 solutions (X, Y) different from (x_0, y_0), where $X \neq Y$ and X, Y are functions of x_0, y_0.

4. Let Ox_1, Oy_1 be rays in the interior of a convex angle xOy such that

$$\angle xOx_1 = \angle yOy_1 < \frac{1}{3}\angle xOy.$$

Points K on Ox_1 and L on Oy_1 are fixed so that $OK = OL$, and points A and B vary on rays Ox and Oy respectively such that the area of the pentagon $OAKLB$ remains constant. Prove that the circumcircle of triangle OAB passes through a fixed point other than O.

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