38-th German Mathematical Olympiad 1999

4-th Round – Rostock, May 2–5

Grade 10

First Day

1. Consider all tetrahedra with side lengths 3cm, 4cm, 5cm, 6cm, $3\sqrt{5}$cm and $2\sqrt{13}$cm.

2. Let be given the number $z = 100000004$ in the base a, where $a \geq 5$ is an integer. Prove that z is not prime.

3. Prove that if the sum of three numbers is 15, then the sum of their squares is at least 75.

Second Day

4. Find all triples (x,y,z) of natural numbers such that

   (i) $x > y > z > 0$ and

   (ii) $1/x + 2/y + 3/z = 1$.

5. Prove that for each quadruple (a,b,c,d) of positive numbers there is a point S inside a given regular tetrahedron $ABCD$, such that the distances from S to the four faces of the tetrahedron are in the ratio $a:b:c:d$.

6. Let ABC be an equilateral triangle and P be a point in the same plane. Prove that $AP \leq BP + CP$ and find a point P distinct from B,C for which equality holds.

Grades 11-13

First Day

1. Find all a) natural numbers; b) integers x,y which satisfy the equality $x^2 + xy + y^2 = 97$.

2. Determine all real numbers x for which

$$1 + \frac{x}{2} - \frac{x^2}{8} \leq \sqrt{1+x} \leq 1 + \frac{x}{2}.$$

The IMO Compendium Group,
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3. A mathematician investigates methods of finding area of a convex quadrilateral obtains the following formula for the area A of a quadrilateral with consecutive sides a, b, c, d:

$$A = \frac{a+c}{2} \cdot \frac{b+d}{2} \ (1) \qquad \text{and} \qquad A = \sqrt{(p-a)(p-b)(p-c)(p-d)} \ (2)$$

where $p = (a+b+c+d)/2$. However, these formulas are not valid for all convex quadrilaterals. Prove that (1) holds if and only if the quadrilateral is a rectangle, while (2) holds if and only if the quadrilateral is cyclic.

Second Day

4. A convex polygon P is placed inside a unit square Q. Prove that the perimeter of P does not exceed 4.

5. Consider the following inequality for real numbers x, y, z:

$$|x-y| + |y-z| + |z-x| \leq a\sqrt{x^2+y^2+z^2}.$$

   (a) Prove that the inequality is valid for $a = 2\sqrt{2}$.

   (b) Assuming that x, y, z are nonnegative, show that the inequality is also valid for $a = 2$.

6A. Suppose that an isosceles right-angled triangle is divided into m acute-angled triangles. Find the smallest possible m for which this is possible.

6B. Determine all pairs (m, n) of natural numbers for which $4^m + 5^n$ is a perfect square.