37-th German Mathematical Olympiad 1998

4-th Round – Potsdam, May 3–6

Grade 10

First Day

1. Prove that, if $n^{n^n}+n^n+n+1$ and n are prime numbers, then $n^{n^n}-n^n+n-1$ is also a prime number.

2. A tetrahedron $ABCD$ is inscribed in a half-sphere of radius r so that A,B,C lie on its basic circle and D lies on the circumference of the half-sphere. Denote by V and u the volume of the tetrahedron and the perimeter of triangle ABC, respectively. Prove that $V \leq \frac{u^3}{324}$. When does equality hold?

3. Andreas experiments with a computer program. The program selects at random 101 integers from 1 to 200 and then finds two of the selected numbers, one of which divides the other. The program worked each time, but Andreas wonders if this is always possible. Brigitte claims that this can be shown without a computer program, by an application of the fact that there are only 100 even numbers in the given range. How can she prove her claim?

Second Day

4. Do there exist three consecutive odd integers whose sum of squares is a four-digit number having all its digits equal?

5. Prove that
$$1998 < 1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{1000000}} < 1999.$$

6. Axel has drawn a triangle ABC, a line parallel to AB which meets AC and BC at F and E respectively, and an arbitrary point D on side AB. He wants to measure the areas of triangles ADF, DBE, DEF and FEC. Ingrid claims that it is possible to determine the area of DEF uniquely, having known the other three areas. How can she achieve that? Give a method and prove that it always works.

Grades 11-13

First Day

1. Find all possible numbers of lines in a plane which intersect in exactly 37 points.

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2. Two pupils A and B play the following game. They begin with a pile of 1998 matches and A plays first. A player who is on turn must take a nonzero square number of matches from the pile. The winner is the one who makes the last move. Decide who has the winning strategy and give one such strategy.

3. For each nonnegative integer k find all nonnegative integers x, y, z such that

$$x^2 + y^2 + z^2 = 8^k.$$

Second Day

4. Let a be a positive real number. Show that the polynomial

$$p(x) = a^3x^3 + a^2x^2 + ax + a$$

has an integer root only for $a = 1$, and find that root.

5. A sequence (a_n) is given by $a_0 = 0$, $a_1 = 1$ and $a_{k+2} = a_{k+1} + a_k$ for all integers $k \geq 0$. Prove that the inequality

$$\sum_{k=0}^{n} \frac{a_k}{2^k} < 2$$

holds for all positive integers n.

6A. Find all real solutions to the system

$$\begin{aligned} x^5 &= 21x^3 + y^3, \\ y^5 &= x^3 + 21y^3. \end{aligned}$$

6B. Prove that the following statement holds for all odd integers $n \geq 3$: If a quadrilateral $ABCD$ can be partitioned by lines into n cyclic quadrilaterals, then $ABCD$ is itself cyclic.