36-th German Mathematical Olympiad 1997

4-th Round – Essen, May 4–7

Grade 10

First Day

1. In a $4 \times n$ set of playing cards, n cards are colored in each of 4 given colors. One noticed that the probability that 5 randomly chosen cards will have the same color increases as n increases: namely, for $n = 8$ this probability is 0.1112%, whereas for $n = 13$ the probability is 0.1981%. Is there a positive integer n for which this probability is as large as 0.5%?

2. Consider 100 rational numbers $Q_i = a_i/b_i$, not all equal, where a_i, b_i are positive integers. Show that $\dfrac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n}$ is greater than the mimimum of Q_i, and smaller than the maximum of Q_i.

3. Let $ABCD$ be a trapezoid with $AB \parallel CD$ and $AB = 2CD$. The diagonals AC and BD intersect at S. A variable line g through S divides the trapezoid into two pieces of areas F_1, F_2, where $F_1 > F_2$.

   (a) Prove that the ratio $v = F_1/F_2$ attains its maximum value for exactly one line g.

   (b) Compute the maximum value of v.

Second Day

4. Consider a table 4×4 whose cells are filled with numbers $1, 2, \dots, 16$ in the usual way. Let us delete eight numbers from the table in such a way that from each row or column exactly two numbers are deleted. Prove that the sum of the remaining numbers is independent on the selection of numbers to be deleted.

5. Prove that for an arbirrary positive integer n there is a positive integer z divisible by n which has exactly two different digits in the decimal representation.

6. At each vertex of a square $ABCD$, a quarter-circle is centered so that it passes through two other vertices of the square. These four quarter-circles intersect each other at points E, F, G, H inside the square. The points E, F, G, H form a smaller square $\mathcal{Q}$. Also, points E, F, G, H determine arcs on the quarter circles which form a curved quadrilateral $\mathcal{V}$. Finally, a circle $\mathcal{K}$ is inscribed in $\mathcal{V}$, touching all its ”sides”. Check if the areas of $\mathcal{Q}$ and $\mathcal{K}$ are equal, and if they are different, decide which one is greater.

Grades 11-13

First Day

1. Prove that there are no perfect squares a, b, c such that $ab - bc = a$.

2. For a positive integer k, let us denote by $u(k)$ the greatest odd divisor of k. Prove that, for each $n \in \mathbb{N}$,
$$\frac{1}{2^n}\sum k = 1^{2^n}\frac{u(k)}{k} > \frac{2}{3}.$$

3. In a convex quadrilateral $ABCD$ we are given that
$$\angle CBD = 10^\circ, \ \angle CAD = 20^\circ, \ \angle ABD = 40^\circ, \ \angle BAC = 50^\circ.$$
Determine the angles $\angle BCD$ and $\angle ADC$.

Second Day

4. Find all real solutions (x,y,z) of the system of equations
$$x^3 = 2y - 1,$$
$$y^3 = 2z - 1,$$
$$z^3 = 2x - 1.$$

5. We are given n discs in a plane, possibly overlapping, whose union has the area 1. Prove that we can choose some of them which are mutually disjoint and have the total area greater than $1/9$.

6A. Let us define f and g by
$$f(x) = x^5 + 5x^4 + 5x^3 + 5x^2 + 1,$$
$$g(x) = x^5 + 5x^4 + 3x^3 - 5x^2 - 1.$$
Determine all prime numbers p such that, for at least one integer x, $0 \le x < p - 1$, both $f(x)$ and $g(x)$ are divisible by p. For each such p, find all x with this property.

6B. An approximate construction of a regular pentagon goes as follows. Inscribe an arbitrary convex pentagon $P_1P_2P_3P_4P_5$ in a circle. Now choose an arror bound $\varepsilon > 0$ and apply the following procedure.

   (a) Denote $P_0 = P_5$ and $P_6 = P_1$ and construct the midpoint Q_i of the circular arc $P_{i-1}P_{i+1}$ containing P_i.

   (b) Rename the vertices $Q_1, \dots, Q_5$ as $P_1, \dots, P_5$.

   (c) Repeat this procedure until the difference between the lengths of the longest and the shortest among the arcs P_iP_{i+1} is less than ε.

   Prove this procedure must end in a finite time for any choice of ε and the points P_i.

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