4-th German Mathematical Olympiad 1965

Final Round

Grade 10

First Day

1. All 30 contestants who take part in a competition are given one of three newly published books which cost 18, 24 or 30 marks, at least one of each. How many ways are there to purchase the 30 books if their total value should be 600 marks?

1. Given a positive parameter p, find all real numbers x satisfying the inequality

$$\frac{x}{p} - \frac{2p}{x} < 2.$$

2. Prove the following statement: If the sum of three natural numbers is a multiple of 6, then so is the sum of their cubes.

Second Day

4. Let M be a point on the hypotenuse AB of a right triangle ABC. Show that $AM^2 \cdot BC^2 + BM^2 \cdot AC^2 = CM^2 \cdot AB^2$.

5. Given a tetrahedron of edge a, consider the plane passing through the midpoint of an edge and parallel to a pair of opposite edges distinct from this edge. Compute the area of this plane inside the tetrahedron.

6. In grades 5 to 8 of a school there are 300 pupils. Denote by a,b,c,d,e,f,g the number of pupils who read magazine *alpha*, *beta*, *gamma*, *alpha* and *beta*, *beta* and *gamma*, *gamma* and *alpha*, all three journals, respectively. How many pupils read none or only one magazine?

Grades 11-13

First Day

1. For a given positive real parameter p, solve the equation

$$\sqrt{p+x} + \sqrt{p-x} = x.$$

2. Determine which of the prime numbers $2,3,5,7,11,13,109,151,491$ divide $z = 1963^{1965} - 1963$.

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3. Two parallelograms $ABCD$ and $A'B'C'D'$ are given in space. Points A'', B'', C'', D'' divide the segments AA', BB', CC', DD' in the same ratio. What can be said about the quadrilateral $A''B''C''D''$?

Second Day

4. Find the locus of points in the plane, the sum of whose distances from the sides of a regular polygon is five times the inradius of the pentagon.

5. Determine all triples of nonzero decimal digits (x,y,z) for which the equality

$$\sqrt{\underbrace{xx x\dots x}_{2n}-\underbrace{yyy\dots y}_{n}}=\underbrace{zzz\dots z}_{n}$$

holds for at least two different natural numbers n.

6. Let α, β, γ be the angles of a triangle. Prove that

$$\cos\alpha+\cos\beta+\cos\gamma\leq\frac{3}{2}$$

and find the cases of equality.

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