40-th German Mathematical Olympiad 2001

4-th Round – Magdeburg, May 13–16

Grade 10

First Day

1. (a) Prove that for each natural number a, a^2 is either of the form $4k$ or of the form $8k+1$ (where $k \in \mathbb{N}$).

   (b) At most, how many equal last decimal digits can a perfect square have?

2. Three pupils spent several hours trying to investigate the inequality

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n^2}>10,$$

   where $n \in \mathbb{N}$. On a meeting, they gave the following statements:

   A: ”The inequality is wrong for every n.“

   B: ”There exists n for which the inequality is true.“

   C: ”More precisely, there exists one such n between 1000^2 and 1050^2.“

   Which of these statements are true, and which are false?

3. Let be given a circle of radius 1 and points Z, A, B on it. We denote by k the arc AB of the circle not containing Z. For every point P on k, point P' on the ray ZP is such that $ZP \cdot ZP' = 4$. Describe and draw the locus of points P'.

Second Day

4. (a) An equilateral triangle ABE is constructed in the exterior of a square $ABCD$ of side length a. The triangle ABE is rotated around point B in the positive direction until point E comes to the position of C; then the triangle is rotated around C in the positive direction until A somes to D, etc. This procedure ends when point E returns to its initial position. How long trajectory did point E describe?

   (b) Let ABI be an equilateral triangle drawn in the interior of the square. The procedure as described in part (a) is applied on triangle ABI until I returns to its initial position. How long is the trajectory of point I?

5. In how many ways can the number 2000 be written as a product of three natural numbers? The representations which differ in the order of factors only, are considered distinct.

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6. Inside a square of side length 12cm are given 20 points, no three of which lie on a line. show that there exists a triangle with vertices in the given points and the area not exceeding 8cm^2.

Grades 11-13

First Day

1. Determine all real numbers q for which the equation $x^4 - 40x^2 + q = 0$ has four real solutions which form an arithmetic progression.

2. Determine the maximum possible number of points inside a 14×28 rectangle, such that each two of the points are at a distance greater than 10.

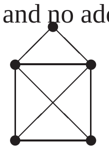
3. Wiebke and Stefan play the following game on a rectangular sheet of paper. They start with a rectangle with 60 rows and 40 columns and cut it in turns into smaller rectangles. The cuttings must be made along the gridlines, and a player in turn may cut only one smaller rectangle. By that, Stefan makes only vertical cuts, while Wiebke makes only horizontal cuts. A player who cannot make a regular move loses the game.

   (a) Who has a winning strategy if Stefan makes the first move?

   (b) Who has a winning strategy if Wiebke makes the first move?

Second Day

4. In how many ways can the ”Nikolaus' House” (see the picture) be drawn? Edges may not be erased nor duplicated, and no additional edges may be drawn.



5. The Fibonacci sequence is given by $x_1 = x_2 = 1$ and $x_{k+2} = x_{k+1} + x_k$ for each $k \in \mathbb{N}$.

   (a) Prove that there are Fibonacci numbesr that end in a 9 in the decimal system.

   (b) Determine for which n can a Fibonacci number end in n 9-s in the decimal system.

6. *(Grade 11)* In a pyramid $SABCD$ with the base $ABCD$ the triangles ABD and BCD have equal areas. Points M,N,P,Q are the midpoints of the edges AB,AD,SC,SD respectively. Find the ratio between the volumes of the pyramids $SABCD$ and $MNPQ$.

6. *(Grades 12-13)* Let ABC be a triangle with $\angle A = 90^\circ$ and $\angle B < \angle C$. The tangent at A to the circumcircle k of $\triangle ABC$ intersects line BC at D. Let E be the reflection of A in BC. Also, let X be the feet of the perpendicular from A to BE and let Y be the midpoint of AX. Line BY meets k again at Z. Prove that line BD is tangent to the circumcircle of $\triangle ADZ$.