29-th German Federal Mathematical Competition 1998/99

Second Round

1. The vertices of a regular $2n$-gon (with $n > 2$ an integer) are labelled with the numbers $1, 2, \ldots, 2n$ in some order. Assume that the sum of the labels at any two adjacent vertices equals the sum of the labels at the two diametrically opposite vertices. Prove that this is possible if and only if n is odd.

2. For every natural number n, let $Q(n)$ denote the sum of the decimal digits of n. Prove that there are infinitely many positive integers k with $Q(3^k) \geq Q(3^{k+1})$.

3. Let P be a point inside a convex quadrilateral $ABCD$. Points K, L, M, N are given on the sides AB, BC, CD, DA respectively such that $PKBL$ and $PMDN$ are parallelograms. Let S, S_1, and S_2 be the areas of $ABCD$, $PNAK$, and $PLCM$. Prove that
$$\sqrt{S} \geq \sqrt{S_1} + \sqrt{S_2}.$$

4. A natural number is called *bright* if it is the sum of a perfect square and a perfect cube. Prove that if r and s are any two positive integers, then

   (a) there exist infinitely many positive integers n such that both $r+n$ and $s+n$ are bright;

   (b) there exist infinitely many positive integers m such that both rm and sm are bright;

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