10-th German Federal Mathematical Competition 1979/80

Second Round

1. Prove that if none of the natural numbers a and b is a perfect cube, then $\sqrt[3]{a}+\sqrt[3]{b}$ is irrational.

2. Let P be a set of n prime numbers, and let M be a set of more than n natural numbers which are not all squares and whose all prime factors are in P. Show that there always exists a nonempty subset T of M whose product of elements is a square.

3. In a triangle ABC, points P,Q,R distinct from the vertices of the triangle are chosen on sides AB,BC,CA, respectively. The circumcircles of the triangles APR, BPQ, and CQR are drawn. Prove that the centers of these circles are the vertices of a triangle similar to triangle ABC.

4. The sequence (a_n) is defined by $a_1=1$, $a_2=2$ and

$$a_{n+2}=\begin{cases} 5a_{n+1}-3a_n & \text{if } a_na_{n+1} \text{ is even,} \\ a_{n+1}-a_n & \text{if } a_na_{n+1} \text{ is odd.} \end{cases}$$

   (a) Prove that the sequence contains infinitely many positive terms and infinitely many negative terms.

   (b) Prove that no term of the sequence equals zero.

   (c) Show that if $n=2^k-1$ for $k=2,3,4,\dots$, then a_n is divisible by 7.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com