

36-th German Federal Mathematical Competition 2005/06

Second Round

1. A circle is divided into $2n$ congruent sectors, half of which are colored black and the other half white. The white sectors are numbered from 1 to n in clockwise direction, while the black sectors are numbered from 1 to n in counterclockwise direction (starting from an arbitrary sector). Show that there are n consecutive sectors containing all numbers from 1 to n .
2. Find all functions $f : \mathbb{Q}^+ \rightarrow \mathbb{R}$ that satisfy the equality

$$f(x) + f(y) + 2xyf(xy) = \frac{f(xy)}{f(x+y)} \quad \text{for all } x, y \in \mathbb{Q}^+.$$

3. A point P is given inside an acute-angled triangle ABC . Let A', B', C' be the orthogonal projections of P on the sides BC, CA, AB , respectively. Determine the locus of points P for which $\angle BAC = \angle B'A'C'$ and $\angle CBA = \angle C'B'A'$.
4. A positive integer is called *digit-reduced* if at most nine different digits occur in its decimal representation. (Leading zeros are omitted.) Let M be a finite set of digit-reduced integers. Prove that the sum of the reciprocals of the elements of M is less than 180.