French IMO Selection Test 2004

First Day

1. Consider the set $A = \{n, n+1, n+2, \dots, n+17\}$, where n is a positive integer. Does there exist n for which set A can be partitioned into two (disjoint) subsets B and C with equal products of elements?

2. In a parallelogram $ABCD$, points M on side AB and N on side BC are taken so that $AM = CN \neq 0$. The lines AN and CM intersect at Q. Prove that DQ bisects the angle ADC.

3. Every point with integer coordinates in the plane is the center of a disk with radius $1/1000$.

   (a) Prove that there exists an equilateral triangle whose vertices lie in different disks.

   (b) Prove that every equilateral triangle with vertices in different disks has side length greater than 96.

Second Day

4. Given a positive integer n, let $a_1, \dots, a_n, b_1, \dots, b_n$ be $2n$ positive numbers such that $a_1 + \cdots + a_n = b_1 + \cdots + b_n = 1$. Find the least possible value of
$$\frac{a_1^2}{a_1+b_1} + \frac{a_2^2}{a_2+b_2} + \cdots + \frac{a_n^2}{a_n+b_n}.$$

5. The incircle of a triangle ABC is tangent to the sides AB, BC, CA at the respective points P, Q, R. Prove that
$$\frac{BC}{PQ} + \frac{CA}{QR} + \frac{AB}{RP} \geq 6.$$

6. Let P denote the set of prime numbers. Consider a subset M of P having at least three elements. Suppose that, for any nonempty and proper finite subset A of M, the prime factors of the number $\prod_{p \in A} p - 1$ belong to M. Prove that $M = P$.

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