French IMO Selection Test 2003

First Day

1. Find the minimum value of $a_1a_2a_3 + b_1b_2b_3 + c_1c_2c_3$ over all permutations $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ of $1, 2, \dots, 9$.

2. An integer point A (i.e. with integer coordinates) in the coordinate plane with origin O is called *invisible* if the segment OA contains an integer point distinct from O and A. Let L be a positive integer. Prove that there exists a square with sides parallel to the coordinate axes such that all integer points inside the square are invisible.

3. For an arbitrary point M inside a triangle ABC, let the line AM intersect the circumcircle of the triangle again at A_1. Find the points M that minimize $\frac{MB \cdot MC}{MA_1}$.

Second Day

4. Let ABC be a triangle and let Γ_1 and Γ_2 be two circles. Suppose that Γ_1 touches AB at B, Γ_2 touches AC at C, and Γ_1 and Γ_2 are externally tangent at D such that $ABDC$ is a convex quadrilateral. Show that the circumcenter of triangle BCD lies on the circumcircle of triangle ABC.

5. Ten cities are connected by one-way air routes in such a way that any city can be reached from any other by several connected flights. Let n be the smallest number of flights needed for a tourist to visit every city and return to the starting city. Number n clearly depends on the flight schedule. Find the greatest possible value of n and a flight schedule yielding this value.

6. Let $p_1, p_2, \dots, p_n$ be distinct primes greater than 3. Show that $2^{p_1p_2\cdots p_n} + 1$ has at least 4^n divisors.

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