French IMO Selection Test 2000

May 31, 2000

1. Four points P, Q, R, S lie on a circle and $\angle PSR$ is right. Let H and K be the projections of Q on the lines PR and PS, respectively. Prove that the line HK bisects the segment QS.

2. A function $f : \mathbb{N} \to \mathbb{N}$ satisfies the following conditions:

   (a) $f(ab) = f(a)f(b)$ for any two coprime positive integers a, b;

   (b) $f(p+q) = f(p) + f(q)$ for any two primes p and q.

   Prove that $f(2) = 2$, $f(3) = 3$, and $f(1999) = 1999$.

3. Let a, b, c, d be four positive numbers with the sum 1. Prove that

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \geq \frac{1}{2},$$

   with the equality if and only if $a = b = c = d = \frac{1}{4}$.

4. Some squares of a 1999×1999 board are occupied by pawns. Find the smallest number of pawns for which it is possible that, for each empty square, the total number of pawns in the row or column with this square is at least 1999.

5. Prove that if A, B, C, D are points on a circle (in this order), then

$$|AB - CD| + |AD - BC| \geq 2|AC - BD|.$$

6. Find all nonnegative integer solutions (x, y, z) of the equation $(x+1)^{y+1} + 1 = (x+2)^{z+1}$.

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