French Mathematical Olympiad 1998

Time: 5 hours.

1. A tetrahedron $ABCD$ satisfies the following conditions: the edges AB, AC and AD are pairwise orthogonal, $AB = 3$ and $CD = \sqrt{2}$. Find the minimum possible value of
$$BC^6 + BD^6 - AC^6 - AD^6.$$

2. Let (u_n) be a sequence of real numbers which satisfies
$$u_{n+2} = |u_{n+1}| - u_n \qquad \text{for all } n \in \mathbb{N}.$$
Prove that there exists a positive integer p such that $u_n = u_{n+p}$ holds for all $n \in \mathbb{N}$.

3. Let $k \geq 2$ be an integer. The function $f : \mathbb{N} \to \mathbb{N}$ is defined by
$$f(n) = n + \left[\sqrt[k]{n + \sqrt[k]{n}}\right].$$
Determine the set of values taken by the function f.

4. Let be given two lines D_1 and D_2 which intersect at point O, and a point M not on any of these lines. Consider two variable points $A \in D_1$ and $B \in D_2$ such that M belongs to the segment AB.
   (a) Prove that there exists a position of A and B for which the area of triangle OAB is minimal. Construct such points A and B.
   (b) Prove that there exists a position of A, B for which the perimeter of triangle OAB is minimal. Show that for such A, B the perimeters of $\triangle OAM$ and $\triangle OBM$ are equal, and that $\dfrac{AM}{\tan \frac{1}{2}\angle OAM} = \dfrac{BM}{\tan \frac{1}{2}\angle OBM}$. Construct such points A and B.

5. Let A be a set of $n \geq 3$ points in the plane, no three of which are collinear. Show that there is a set S of $2n - 5$ points in the plane such that, for each triangle with vertices in A, there exists a point in S which is strictly inside that triangle.

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