French Mathematical Olympiad 1997

Time: 5 hours.

1. Each vertex of a regular 1997-gon is labeled with an integer, so that the sum of the integers is 1. We write down the sums of the first k integers read counterclockwise, starting from some vertex ($k = 1, 2, \ldots, 1997$). Can we always choose the starting vertex so that all these sums are positive? If yes, how many possible choices are there?

2. A region in space is determined by a sphere with center O and radius R, and a cone with vertex O which intersects the sphere in a circle of radius r. Find the maximum volume of a cylinder contained in this region, having the same axis as the cone.

3. Let C be a unit cube and let p denote the orthogonal projection onto the plane. Find the maximum area of $p(C)$.

4. In a triangle ABC, let a, b, c be its sides and m, n, p be the corresponding medians. For every $\alpha > 0$, let $\lambda(\alpha)$ be the real number such that

$$a^\alpha + b^\alpha + c^\alpha = \lambda(\alpha)^\alpha (m^\alpha + n^\alpha + p^\alpha).$$

   (a) Compute $\lambda(2)$.

   (b) Find the limit of $\lambda(\alpha)$ as α approaches 0.

   (c) For which triangles ABC is $\lambda(\alpha)$ independent of α?

5. Given two distinct points A, B in the plane, for each point C not on the line AB we denote by G and I the centroid and incenter of the triangle ABC, respectively.

   (a) For $0 < \alpha < \pi$, let Γ be the set of points C in the plane such that $\angle\left(\overrightarrow{CA}, \overrightarrow{CB}\right) = \alpha + 2k\pi$ as an oriented angle, where $k \in \mathbb{Z}$. If C describes Γ, show that points G and I also describe arcs of circles, and determine these circles.

   (b) Suppose that in addition $\pi/3 < \alpha < \pi$. For which positions of C in Γ is GI minimal?

   (c) Let $f(\alpha)$ denote the minimal GI from the part (b). Give $f(\alpha)$ explicitly in terms of $a = AB$ and α. Find the miminum value of $f(\alpha)$ for $\alpha \in (\pi/3, \pi)$.

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